We select the functions p_i in the form

$$p_1 = -d\Pi/dx_1, \quad p_2 = -mx_2 \tag{3.13}$$

Then, as follows from (3.11)−(3.13), $u_* = -x_2(t)$, and the extremals $x_*(t)$ must be, as before, solutions of the Cauchy problem (3.7). The satisfaction of the transversality conditions (3.9) is guaranteed, if we assume $\lambda_1 = -d\Pi[x_1(t_1)]/dx_1(t_1)$, $\lambda_2 = -mx_2(t_1)$. For the completion of the solution of this problem by the methods of the classical calculus of variations it is further necessary to prove that the control $u_* = -x_2$ obtained provides the minimum of the functional (3.5). This had been proved using Theorem 1.

The examples considered here show the effectiveness of the proposed design of optimal control of the motion of mechanical systems based on the use of the first integrals.

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OPTIMAL CONTROL OF STEPWISE PROCESSES WITH PERIODIC CHARACTERISTICS*

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The problem of optimal control of a stepwise Markov process with periodic characteristics that is not discontinuous with respect to probability is solved. The sufficiency of periodic Markov control strategies is proved, the optimality equation is obtained, and examples of the solution of practical problems are given.

The construction of optimal strategies for the control of stochastic processes is a pressing practical problem. /1–10/. Besides stochastically continuous /1, 2, 5, 7–9/ and purely discontinuous /3, 4, 6/ models of controllable processes, problems in which the controllable stochastic process has a mixed character are of interest. In /1–10/ models with diffusion and intermittent components, and also with other interacting Markov processes were studied. One of the varieties of such combined models, including a chain with discrete time and a stochastically continuous intermittent process are considered in this paper. Problems of the optimal control of such system were investigated in /10/ in a finite time interval. Here the problem of synthesis in an infinite time interval is considered on the assumption that all the characteristics of the controlled model are periodic time functions.

1. Notation and definitions.

A two-component Markov intermittent stochastic process (ξ_t, ψ_t) is considered here in an infinite time interval $I = [0, \infty)$. The component ξ_t represents a stochastically continuous process, the jumps of the component ψ_t appear at known instants $\tau, 2\tau, \ldots$. We denote by X the space of component ξ_t, and Y is the space of component states ψ_t that are finite or denumerable sets. The term state of the process (ξ_t, ψ_t) at the instant

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$t \in I$ is understood to be the ordered pair $(x, y) \in X \times Y$. The trajectories are understood to be infinite to the right and have a limit on the left.

Assume that each pair $(x, y) \in X \times Y$ has a Borel set of admissible controls $A(x, y)$ specified. If at the instant t the process (ξ_t, ψ_t) takes the value (x, y), and the control $a \in A(x, y)$ is specified, then $\lambda_{x,z}(a, y, t)$ is the rate of transition of the component ξ_t from the state x to state $z \in X$ at instant t. In that case the rate of payoff at the instant t is $e^{-\alpha t} r_t(x, y, a)$. Here and below $\alpha > 0$ is the discounting coefficient. Below, only periodic models for which $\lambda_{x,z}(a, y, t+\tau) = \lambda_{x,z}(a, y, t)$ and $r_{t+\tau}(x, y, a) = r_t(x, y, a)$ are considered.

If $(\xi_{n\tau-0}, \psi_{n\tau-0}) = (x, y)$ and the control $a \in A(x, y)$ is selected, then $P_{y,z}(a, x)$ is the probability of the component ψ_t passing from state y to state $z \in Y$. In that case the payoff at the instants $n\tau$ is $e^{-\alpha n\tau} R(x, y, a)$.

Definition. The set $Z = \{I, X, Y, A, \lambda, r, P, R\}$, where $A = \bigcup_{(x,y)\in X\times Y} A(x, y)$, is called a model.

We denote by $(x, y)_0^t$ the trajectory of process (ξ_t, ψ_t) in the interval $[0, t] \subset I$.

Definition. The measurable mapping π, is called the strategy of the control process (ξ_t, ψ_t) which puts each trajectory $(x, y)_0^t$ in correspondence with the point $a = \pi[(x, y)_0^t] \in A(x_t, y_t)$, where (x_t, y_t) is the state of the process at instant t. If the control depends only on time and the final state $\pi[(x, y)_0^t] = \varphi(t, x_t, y_t)$ the strategy is called Markov strategy. The Markov strategy (MS) is called periodic, if $\varphi(t+\tau, x, y) = \varphi(t, x, y)$. For a fixed strategy π we use the notation $a_t = \pi[(x, y)_0^t]$.

Definition. The average payoff is called the estimate of the strategy π in the interval I

$$\omega_0(x, y, \pi) = M^{\pi}_{(x,y)} \left\langle \int_0^{\infty} e^{-\alpha t} r_t(\xi_t, \psi_t, a_t)\, dt + \sum_{n=1}^{\infty} e^{-\alpha n\tau} R(\xi_{n\tau-0}, \psi_{n\tau-0}, a_{n\tau-0}) \right\rangle \tag{1.1}$$

where (x, y) is the initial state of the process at the instant $t = 0$, $M^{\pi}_{(x,y)}$, which is the symbol of the expectation with respect to the measure $P^{\pi}_{(x,y)}$ in the space of trajectories of the process (ξ_t, ψ_t) which begins from the state (x, y) for a fixed control strategy π. The quantity

$$v_0(x, y) = \sup_{\pi} \omega_0(x, y, \pi) \tag{1.2}$$

is called the estimate of the model Z. Besides model Z we shall consider a "derived model" $Z_t^T(S)$ specified in the interval $I' = [t, T]$ with the final payoff $S: X \times Y \to \mathbf{R}^1$. Evaluation of the strategy and the model for $Z_t^T(S)$ is henceforth denoted by the symbols $\omega_t^T(x, y, \pi, S)$ and $v_t^T(x, y, S)$. If $T = \infty$ and $S = 0$, these arguments are omitted.

Definition. The strategy π is called ε-optimal, if $\omega_0(x, y, \pi) \geqslant v_0(x, y) - \varepsilon$ for all $(x, y) \in X \times Y$, the 0-strategy is called optimal.

We shall assume that the following conditions are satisfied.

1°. The functions $\lambda_{x,z}(a, y, t)$ is measurable and uniformly bound with respect to all arguments.

2°. $\sum_{z\in X} \lambda_{x,z}(a, y, t) = 0$

3°. $\| \sup_{a\in A(x,y)} |r_t(x, y, a)| \| + \| \sup_{a\in A(x,y)} |R(x, y, a)| \| < \infty$

Here and below

$$\| f_t(x, y) \| = \sup_{t\in[0,\tau),\,(x,y)\in X\times Y} |f_t(x, y)|;\ \| f(x, y) \| = \sup_{(x,y)\in X\times Y} |f(x, y)|$$

2. Fundamental theorems. We introduce the following notation:

$$D^a g_t(x, y) = e^{-\alpha t} r_t(x, y, a) + \sum_{z\in X} \lambda_{x,z}(a, y, t) g_t(z, y)$$

$$E^a g(x, y) = e^{-\alpha\tau} \left[R(x, y, a) + \sum_{z\in Y} P_{y,z}(a, x) g(x, z) \right]$$

where g is a real function in $[0, \tau) \times X \times Y$ or $X \times Y$, respectively.

The basic mathematical results of the present paper are covered by the following two theorems.

Theorem 1. For any $\varepsilon > 0$ there exists a periodic ε-optimal strategy. If $A(x, y)$ are compacta, the functions r and R are upper semicontinuous in a, and the functions λ and P are continuous in a, an optimal periodic strategy exists.

Theorem 2. a) The evaluation of the model $v_t(x, y)$ for $t < \tau$ is the unique absolutely continuous solution of the equation of optimality

$$v_t(x,y)=\sup_{a\in A(x,y)} E^a v_0(x,y)+\int_t^{\tau}\sup_{a\in A(x,y)} D^a v_\theta(x,y)\,d\theta \tag{2.1}$$

b) $v_{t+\tau}(x,y)=e^{-\alpha\tau}v_t(x,y)$

c) the periodic strategy φ^* is optimal, if and only if for all $(x,y)\in X\times Y$ we have

$$(d/dt+D^{a_t^*})\,v_t(\xi_t,\psi_t)=0,\quad dt\times P^{\varphi^*}_{(x,y)} - \text{p.c.} \tag{2.2}$$

$$E^{a^*_{\tau-0}}v_0(\xi_{\tau-0},\psi_{\tau-0})=v_{\tau-0}(\xi_{\tau-0},\psi_{\tau-0}),\quad P^{\varphi^*}_{(x,y)} - \text{p.c.} \tag{2.3}$$

When solving problems of optimal control Theorem 1 enables us to confine the investigation to the class of periodic strategies. The numerical construction of optimal periodic strategies is made possible by Theorem 1.

To prove the theorems we require the following auxiliary constructions.

Let $\Phi(y)$ be the set of all measurable function in $A(x,y)$ from $[0,\tau)\times X$. Each MS is, obviously, specified by the sequence $[f_0,f_1,\ldots]$, where f_n is the mapping which puts some element $f_n(y)\in\Phi(y)$ in correspondence to each $y\in Y$. The symbol φ^n will be used to denote the sequence $[f_n,f_{n+1},\ldots]$, when $\varphi=[f_0,f_1,\ldots]$.

Let f be the mapping of set Y in $\Phi=\bigcup_{y\in Y}\Phi(y)$, such that $f(y)\in\Phi(y)$ and $u\colon X\times Y\to\mathbf{R}^1$ is some uniformly bounded function. We shall denote by $L(f)\,u$ (when $t=0$) the solution of the following Cauchy problem:

$$(d/dt+D^{f(y)(t,x)})\,g_t(x,y)=0 \tag{2.4}$$

$$g_\tau(x,y)=E^{f(y)(\tau-0,x)}\,u(x,y) \tag{2.5}$$

In addition to the operator $L(f)$ we shall consider the operator U specified by the formula

$$Uu=\sup_{f\colon Y\to\Phi}L(f)\,u \tag{2.6}$$

Lemma 1. For any $f:Y\to\Phi$

a) if $u_1\geqslant u_2$, then $L(f)\,u_1\geqslant L(f)\,u_2$ and $Uu_1\geqslant Uu_2$,

b) $L(f)(u+c)=L(f)\,u+e^{-\alpha\tau}c$, and $U(u+c)=Uu+e^{-\alpha\tau}c$, where c is an arbitrary constant function specified in $X\times Y$;

c) the operators $L(f)$ and U are compressive, and

$$\|L(f)\,u_1-L(f)\,u_2\|\leqslant e^{-\alpha\tau}\|u_1-u_2\|;\quad \|Uu_1-Uu_2\|\leqslant e^{-\alpha\tau}\|u_1-u_2\|;$$

d) for any $\varepsilon>0$ and any function $u(x,y)$ a mapping $f:Y\to\Phi$ exists such that $L(f)\,u\geqslant Uu-\varepsilon$.

Proof. Note that the matrix $\exp\left(\int_t^{\tau}\Lambda(f,\ y,\ \theta)\,d\theta\right)$ is a stochastic matrix /11/. Hence the statements a) and b) above follow directly from (2.4)–(2.6). Proof of c) above is identical with that in /3/.

We set

$$h(x,y)=\sup_{a\in A(x,y)}E^a u(x,y) \tag{2.7}$$

It follows from the results obtained in /5/ that $Uu(x,y)=v_0{}^\tau(x,\ y,\ h)$. It was also proved there that for any fixed y there exists in model $Z_0{}^\tau(h)$ and $\varepsilon/2$–optimal MS $f(y)(t,x)$. Moreover, in accordance with (2.7) we have $\exists a^*(x)=f(y)(\tau-0;x)\colon E^{a^*}u>h-\frac{\varepsilon}{2}$. Repeating this reasoning for all $y\in Y$ we obtain the mapping $f\colon Y\to\Phi$. To prove the inequality $L(f)\,u\geqslant Uu-\varepsilon$ it is sufficient to note that $\omega_0{}^\tau(x,y,f(y),h)$ is the unique solution of (2.4) with initial condition $g_\tau=h$ (see /5/). The lemma is proved.

Lemma 2. Let $\varphi=[f_0,f_1,\ldots]$ be an arbitrary MS. Then

a) $\omega_{n\tau}(x,y,\varphi)=e^{-\alpha n\tau}\,\omega_0(x,y,\varphi^n)$

b) $\omega_0(\varphi)=L(f_0)\ldots L(f_n)\,\omega_0(\varphi^{n+1})$

Proof. Statement a) above follows from (1.1) and the periodicity of the function λ, r. According to /5/, $\omega_t{}^\tau(x,y,\varphi,\omega_{\tau-0})$ is the unique solution of (2.4) for $f=f_0$ with the initial condition $g_\tau(x,y)=\omega_{\tau-0}(x,y,\varphi)$. From a) it follows that $\omega_0(\varphi)=L(f_0)\,\omega_0(\varphi^1)$. Continuing this reasoning, we obtain the required $\omega_0(\varphi)=L(f_0)\ldots L(f_n)\,\omega_0(\varphi^{n+1})$.

Lemma 3. Let u^* be a stationary point of the operator U. Then for any $f\colon Y\to\Phi$ $L^n(f)\,u^*\xrightarrow[n\to\infty]{}\omega_0(f^\infty)$. Here and below f^∞ is the periodic strategy $\varphi=[f,f,\ldots]$.

Proof. According to Lemma 2 $\omega_0(f^\infty) = L(f)\,\omega_0(f^\infty)$. Using c) of Lemma 1, we obtain $\| L(f) \ldots L(f)\, u^* - \omega_0(f^\infty) \| \leqslant e^{-\alpha(n+1)\tau} \| \omega_0(f^\infty) - u^* \| \underset{n\to\infty}{\longrightarrow} 0$, which it was required to prove.

Proof of Theorem 1. Let us prove that for any $\varepsilon > 0$ in the model this is an ε-optimal MS. It follows from condition 3^0 that for any $\delta > 0$ there exists an N such that for $n \geqslant N$ the inequality

$$e^{-\alpha n\tau}\Big\{\frac{1}{\alpha}(\delta + \| \sup_{a\in A(x,y)} | r_t(x,y,a) | \|) + \frac{1}{1-e^{-\alpha\tau}}(\delta + \| \sup_{a\in A(x,y)} | R(x,y,a) | \|)\Big\} \leqslant \frac{\varepsilon}{3} \tag{2.8}$$

holds. It is well-known* (*Piunovskii A.B. Optimal control of a continuously discrete intermittent Markov process with complete information. No.4512-80, dep. in VINITI, 27.10.80. Moscow, 1980.) that in model $Z_0^{N\tau}(0)$ there exists an $\varepsilon/3$ optimal MS $\varphi_1(t, x, y)$. Finally, for $t \geqslant N\tau$ we select the strategy $\varphi_2(t, x, y)$ so that the inequalities are satisfied.

$$r_t(x, y, \varphi_2(t, x, y)) \geqslant - \sup_{a\in A(x,y)} | r_t(x, y, a) | - \delta \tag{2.9}$$

$$R(x, y, \varphi_2(n\tau - 0, x, y)) \geqslant - \sup_{a\in A(x,y)} | R(x, y, a) | - \delta$$

Using the definitions (1.1) and (1.2) and inequalities (2.8) and (2.9) it can be readily shown that the MS $\varphi(t, x, y)$, formed by functions φ_1 and φ_2, is ε-optimal.

To prove the sufficiency of periodic strategies we note that $v_0(x, y) \leqslant u^*(x, y)$, where u^* is the stationary point of the operator U. Indeed, by virtue of c) of Lemma 1 and b) of Lemma 2 we have $\omega_0(\varphi) \leqslant L(f_0) \ldots L(f_n)\, u^* + e^{-\alpha(n+1)\tau}(\| \omega_0(\varphi) \| + \| u^* \|) \leqslant u^* + e^{-\alpha(n+1)\tau}(\| u^* \| + \| \omega_0(\varphi) \|) \underset{n\to\infty}{\longrightarrow} u^*$. Let $L(f)\, u^* \geqslant Uu^* - \varepsilon'$, where $\varepsilon' = \varepsilon(1 - e^{-\alpha\tau})$. The existence of the mapping f follows from c) of Lemma 1. Using statements a) and b) of Lemma 1, the validity of inequalities $L^n(f)\, u^* \geqslant u^* - \varepsilon$ is readily proved by methods of complete mathematical induction. By passing to the limit, we obtain $\omega_0(f^\infty) \geqslant u^* - \varepsilon \geqslant v_0 - \varepsilon$, as required. Here Lemma 3 was used.

The second part of the theorem is proved similarly.

Proof of Theorem 2. We set $h_{n\tau}(x, y) = e^{-\alpha(n-1)\tau} \sup_{a\in A(x,y)} E^a \times v_0(x, y)$. According to Lemma 2, when $t < \tau$, we have $v_t(x, y) = v_t^\tau(x, y, h_\tau)$. The integral Eq.(2.1) for model $Z_t^\tau(h_\tau)$ follows directly from the results obtained in /5/. The integral representations for $v_{\tau+t} = v_{\tau+t}^{(n+1)\tau}(x, y, h_{(n+1)\tau})$ and $v_t(x, y) = v_t^{n\tau}(x, y, h_{n\tau})$ when $(n-1)\tau \leqslant t < n\tau$ are obtained similarly. Statement b) is checked by substitution; the existence and uniqueness of a solution of integral Eq.(2.1) and similar ones are proved by the standard method of compressive mappings. The proof of c) is carried out using a) of Lemma 2 and the following identity:

$$\omega_t(x, y, \pi) = F_t(x, y) + \mathrm{M}^\pi_{(x,y)}\Big\langle \omega_{\tau-0}(\xi_{\tau-0}, y, \pi) - F_{\tau-0}(\xi_{\tau-0}, y) + \int_t^\tau \{d/d\theta + D^{a_\theta}\}\, F_\theta(\xi_\theta, y)\, d\theta \Big\rangle \quad (t < \tau)$$

which holds by virtue of conditions 1, and 2 for any strategy π and arbitrary absolutely continuous function $F_t(x, y)$ on $[0, \tau) \times X \times Y$. For the function $F_t(x, y)$ the solution of (2.1) must be taken.

3. Examples. *The problem of controllable subsystem.* Consider the divice capable of servicing requirements of two types stored beforehand in bunkers 1 and 2. Interruption of started servicing is forbidden, and the selection of recurrent demand is carried out by the person operating the control device. The time of servicing of the demand of any type is assumed to be exponentially distributed with parameter λ. The guidance of the device operator by a higher-order subsystem consists of the following. At the beginning of each interval $[0, \tau]$, $[\tau, 2\tau]$; ... an indication is received of which of the demands are to be served first (i.e. priority is assigned $y = 1$ or $y = 2$); at the end of the corresponding interval the device operator receives the payoff R or is punished by the penalty r, depending on whether he has been serving a more or less priority demand. Let $P_1 = 0.5 + q$ and $P_2 = 0.5 - q$ be the assignment probabilities of priority 1 or 2, respectively $(-0{,}5 \leqslant q \leqslant 0{,}5)$. For given $\lambda, \tau, R, r, q, \alpha$ (the discount coefficient) it is required to determine the optimal behaviour of the operator of the control device, as the cumulative discounted payoff in an infinite interval.

In accordance with accepted notation $X = \{1, 2\}$; $Y = \{1, 2\}$, where $\xi_t = x \in X$ denotes that the device is occupied by servicing demands of type x, and $\psi_t = y \in Y$ indicates that type y requirements have a higher priority. Let $A(x, y) = \{0, 1\}$, where $a = 1$ $(a = 0)$, indicates a decision of switching (not switching) on the other bunker has been taken. Let $\lambda_{x,z}(a, y, t) = (-1)^{x+z+1} a\lambda$;

$$r_t(x, y, a) = 0; \quad R(x, y, a) = \begin{cases} R, & x = y, \\ -r, & x \neq y; \end{cases} \quad P_{y,z}(a, x) = P_z$$

The equation of optimality (2.1) may be written in the form

$$d/dtv_t(x, y) = -\sup_{a\in\{0,1\}} \{a\lambda [v_t(x', y) - v_t(x, y)]\} \quad (3.1)$$

$$v_{\tau-0}(x, y) = e^{-\alpha\tau}[R_{x,y} + P_1 v_0(x, 1) + P_2 v_0(x, 2)] \quad (3.2)$$

$$x' = \begin{cases} 1, & x=2 \\ 2, & x=1 \end{cases}; \quad R_{x,y} = \begin{cases} R, & x=y \\ -r, & x\neq y \end{cases}$$

Assume that

$$v_{\tau-0}(1, 1) \geqslant v_{\tau-0}(2, 1), \; v_{\tau-0}(1, 2) \leqslant v_{\tau-0}(2, 2) \quad (3.3)$$

It is possible to prove that in this case the solution of system (3.1) has the form $v_t(x, y) = v_{\tau-0}(x, y)\ (x = y)$; $v_t(x, y) = e^{-\lambda(\tau-t)}[v_{\tau-0}(x, y) - v_{\tau-0}(x', y)] + v_{\tau-0}(x', y)\ (x \neq y)$. Substituting the expressions obtained for $t = 0$ into (3.2) and noting that for $x \neq y$ we have $v_{\tau-0}(x, y) = v_{\tau-0}(x, y') - e^{-\alpha\tau}(R + r)$, we obtain a system of linear equations for $v_{\tau-0}$ (1,1) and $v_{\tau-0}$ (2,2), from which $v_{\tau-0}(1, 1) = \Delta_1/\Delta$; $v_{\tau-0}(2, 2) = \Delta_2/\Delta$. This reasoning holds only when inequalities (3.3) equivalent to the inequality $|\Delta_1 - \Delta_2| \leqslant e^{-\alpha\tau}(R + r)\Delta$ are satisfied, whose solution has the form

$$(\alpha + \lambda)\tau \geqslant \ln(1 + 2|q|) \quad (3.4)$$

The optimal strategy in this case is as follows:

$$\varphi(t, x, y) = \begin{cases} 0, & x=y \\ 1, & x\neq y \end{cases}$$

Thus when (3.4) is satisfied, the optimal strategy directs the fulfilment of incoming dispositions.

If instead of (3.3), the following hypotheses are considered:

$$v_{\tau-0}(1, 1) \geqslant v_{\tau-0}(2, 1); \quad v_{\tau-0}(1, 2) \geqslant v_{\tau-0}(2, 2) \quad (3.5)$$

$$v_{\tau-0}(1, 1) \leqslant v_{\tau-0}(2, 1); \quad v_{\tau-0}(1, 2) \leqslant v_{\tau-0}(2, 2) \quad (3.6)$$

then, using similar reasoning, we obtain the optimal strategies

$$\varphi(t, x, y) = \begin{cases} 0, & x=1 \\ 1, & x=2 \end{cases}; \quad \varphi(t, x, y) = \begin{cases} 0, & x=2 \\ 1, & x=1 \end{cases}$$

resepctively, and (3.5) is equivalent to the inequalities $q \geqslant 0$; $(\alpha + \lambda)\tau \leqslant \ln(1 + 2q)$, and (3.6) is equivalent to inequalities $q \leqslant 0$; $(\alpha + \lambda)\tau \leqslant \ln(1 - 2q)$.

Consequently, the optimal behaviour of the control device has the form shown in Fig.1, where region a corresponds to the solution "accept for servicing only requests of the first type", region b to "carry out the incoming orders", and region c "to accept for servicing only a request of the second type". Line ACB of switching is given by the equation $(\alpha + \lambda)\tau = \ln(1 + 2|q|)$.

One channel SMS with refusals. Consider a single-channel queuing system (SQS) into which a Poisson stream enters at a rate $\lambda(t) = b + d\sin^2(\pi t/\tau)$. Let us assume that the SQS has two modes of operation characterized by the rates of servicing μ_1 and μ_2 with $\mu_1 < \mu_2$. The rate of loss related to servicing are r_1 and r_2 in the first and second modes, respectively. Losses related to demand result in a penalty R. It is required to construct the optimal strategy of the system control, i.e. to show for each instant of time the best mode of operation of the SQS.

The mathematical model of the system is the stochastically continuous controlled Markov process with periodic characteristics, which is a special case of the discretely continuous model investigated.

In confirmity with the accepted notation $X = \{1, 2\}$, and $\xi_t = 1$ indicates that the channel is free and $\xi_t = 2$ that it is occupied. The control space $A = \{1, 2\}$ consists of two elements: 1 the first mode of operation, and 2 the second. Component ψ_t does not appear, i.e. $Y = \{1\}$; $R(x, y, a) = 0$) and subsequently the argument y is omitted. The infinitesimal matrix is

$$\Lambda(a, t) = \begin{Vmatrix} -b - d\sin^2\left(\frac{\pi t}{\tau}\right) & b + d\sin^2\left(\frac{\pi t}{\tau}\right) \\ \mu_a & -\mu_a \end{Vmatrix}$$

and the rate of payoff is

$$r_t(x, a) = \begin{cases} 0, & x=1 \\ -r_a - R(b + d\sin^2(\pi t/\tau)), & x=2 \end{cases}$$

The problem was solved for the following values of the parameters: $\tau = 1, \alpha = 2, r_1 = 20, r_2 =$

120, $R = 2$, $\mu_1 = 20$, $\mu_2 = 100$, $b = 2$, $d = 10$. The initial approximation for $v_0(x)$ was assumed to be zero. Equation (2.1) was solved by the iterative method, and the estimate of model $v_0(x)$ was computed with an accuracy of 0.01. Curves of the estimate of the model and of the optimal control strategy are shown in Fig.2. It will be seen that the second mode of SQS operation should be selected in the interval (0, 3; 0,6]. This is explained by the increase in the intensity of the incoming stream. Subsequently in the intervals /1, 2/, /2,3/;... the strategy is periodically repeated.

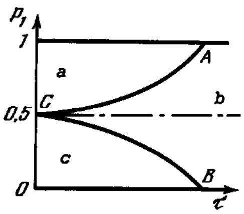

Fig.1

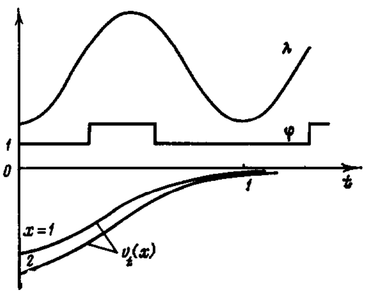

Fig.2

4. Special cases. We shall consider some special cases of the general model of a controllable system defined in Sects.1 and 2. Let $m(X) = 1$; $r_t(x, y, a) = 0$. Here and below $m(D)$ is the power of the final set D. The model investigated is converted into a discounted Markov process of taking decisions /3/. In this case statements appearing in Sect.2 agree with known results.

On the other hand, if we set $m(Y) = 1$; $R(x, y, a) = 0$, the property of stochastic continuity is restored. Such models were investigated on the assumption that $\lambda_{x,z}(a, t) \equiv \lambda'_{x,z}(a)$; $r_t(x, a) \equiv r(x, a)$. It was shown in /7/ that in such models it is sufficient to restrict the investigation to stationary SQS $\varphi(t, x) \equiv \varphi(t)$. This is in good agreement with Theorem 1, if one notes that any arbitrary real number may be taken as τ.

Assume that $m(Y) = 1$; $R(x, y, a) = 0$ but the process is not stationary. The model considered is an example of an important special case of controllable stochastically continuous intermittent Markov processes, namely of a model with periodic functions $\lambda_{x,z}(a, t)$ and $r_t(x, a)$. This enables us to state that in such models it is sufficient to limit the investigation to periodic strategies with the same period τ. According to b) of Theorem 2 the evaluation of the model is a periodic exponentially damped function.

The model investigated as $\tau \to \infty$ converts into a stochastically continuous discounted controllable Markov process. The concepts of Markov and periodic control strategies merge in the statements formulated in Sect.2, and agree with the results obtained in /5/.

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ON UNIFORM LINEAR INVARIANT RELATIONS OF THE EQUATIONS OF DYNAMICS*

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In a tangential stratification of the configuration manifold of a mechanical system, the submanifolds of its trajectories specified in local coordinates by equations that are linear and homogeneous, with respect to velocities, are discussed. The local conditions for the existence of some of such submanifolds in a structural form are established. The results obtained are illustrated by examples taken from solid dynamics.

1. Let $q \in R^n$ be the Lagrangian coordinates of a holonomic mechanical system, $T = {}^1/_2 (a(q)q^\cdot, q^\cdot)$ its kinetic energy, and $F(q) \in R^n$ the generalized force. The equation of motion can be written in a form which can resolve in terms of accelerations,

$$q^{\cdot\cdot} = -(\Gamma q^\cdot, q^\cdot) + F \tag{1.1}$$

or, when the velocity field $q^\cdot = f(q)$ is specified.

$$(f, \nabla) f = F \tag{1.2}$$

Here $\Gamma(q)$ is the connectivity object (see /1/), ∇ denotes covariant differentiation, $(\xi, \eta) = \xi_i \eta^i$; the repeating index is understood to mean summation from 1 to n.

Definition. The relations

$$\varphi_1(q, q^\cdot) = 0, \ldots, \varphi_m(q, q^\cdot) = 0 \quad (m \leqslant 2n) \tag{1.3}$$
$$\operatorname{rank} \| \partial\varphi/\partial q, \quad \partial\varphi/\partial q^\cdot \| = m$$

form, in a certain domain of variation of the variables q and $q^\cdot$ an invariant ensemble for the system of differential equations $q^{\cdot\cdot} = G(q, q^\cdot) \in R^n$, if for each $\alpha = 1, \ldots, m$ the expression

$$\frac{d\varphi_\alpha}{dt} = \left(\frac{\partial\varphi_\alpha}{\partial q}, q^\cdot\right) + \left(\frac{\partial\varphi_\alpha}{\partial q^\cdot}, G\right)$$

has the form

$$\frac{d\varphi_\alpha}{dt} \equiv \sum_{\beta=1}^{m} \varkappa_{\alpha\beta}(q, q^\cdot)\varphi_\beta \tag{1.4}$$

($\varkappa_{\alpha\beta}$ are the continuous functions).

In the tangential destratification TM of the configuration manifold M of a mechanical system, Eqs.(1.3) define locally a certain submanifold. Under conditions (1.4), the integral curve of the equations of motion, which has a common point with this submanifold, lies on it, i.e. the given submanifold is integral.

Let us consider the question of the existence, for Eqs.(1.1), of an ensemble of invariant relations of the form

$$\langle\lambda_{n-m+1}, q^\cdot\rangle = 0, \ldots, \langle\lambda_n, q^\cdot\rangle = 0 \quad (m \leqslant n-1) \tag{1.5}$$

where the vectors $\{\lambda_\alpha(q)\}$ are linearly indepenent of each point, $\langle\xi, \eta\rangle = (a\xi, \eta)$. In doing so, we shall confine ourselves to studying two extreme cases: $m = n-1$ and $m = 1$.

Theorem 1. Let $F \neq 0$. An ensemble of the $(n-1)$–th invariant relations (1.5) exists if and only if the lines of force are geodesic lines of the Riemann manifold $(M, \langle\,,\rangle)$. For $F \equiv 0$, the system has ∞^n of such invariant ensembles.

This theorem is a corollary of Theorem 3 proved below. For $n = 2$, it is given in /2/.

For $F = \operatorname{grad} U(q)$, the condition of the theorem is written analytically as

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